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Solvable Models of the Fokker–Planck Equation: An Approach Based on the Gel'fand–Levitan Method

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From a given solvable Fokker–Planck equation one can construct a number of other solvable models for diffusion in a stable or bistable potential fields using the Gel'fand–Levitan method of the inverse scattering theory. The simplest way of achieving this is to change the lowest eigenvalue and/or the normalization of the lowest eigenfunction of the ordinary differential equation obtained by separating the time-dependent part. For these cases it is shown that the new probability distribution is expressible in terms of integrals involving the original probability distribution and the Gel'fand–Levitan kernel. The possibility of changing the lowest eigenvalue enables one to find bistable potential fields which would correspond to a well-defined long time relaxation rate for the probability current.

KEY WORDS: Diffusion in a bistable potential field; solvable models of the Fokker–Planck equation; the Gel'fand–Levitan method.

1. INTRODUCTION

One of the most remarkable developments in mathematical physics over the past few years has been the application of the inverse scattering theory to solve certain nonlinear partial differential equations arising in different branches of physics. These equations are two dimensional with time as one of the variables and a spatial coordinate as the other. Among other important two-dimensional partial differential equations of physics is the Fokker–Planck equation, which is usually linear and generally has coordinate-dependent coefficients.[(1)] This equation also has been the subject of extensive studies recently, and several exactly solvable models appropriate for a variety of physical problems have appeared in the literature.[(2–7)] Of

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particular interest have been models for diffusion in bistable symmetric or asymmetric potential wells. The latter problem, i.e., diffusion in an asymmetric potential field, turns out to be more difficult to solve in closed form, and even the solvable ones, because of the lack of symmetry, have a more complicated analytic structure. But it is for the construction of solvable models with asymmetric wells that the method of the inverse scattering theory works best, since it enables one to determine a potential field with needed asymmetry, and with a prescribed long-time relaxation rate. In Section 2, following van Kampen's approach, we show how the solution of the Fokker–Planck equation can be expressed in terms of the solution of the Schrödinger equation with a confining potential. Limiting our attention to those cases where the latter equation is exactly solvable, we observe that the potential, and hence the eigenfunctions of the Schrödinger equation, can be changed by changing the normalization and or the energy levels of one or a number of low-lying states with the help of the Gel'fand–Levitan equation.[8–10] The resulting set of wave functions and the potential will correspond to a different, yet solvable, Fokker–Planck equation. In fact it is possible to relate the new probability distribution of the transformed Fokker–Planck equation to the old probability distribution, i.e., the solution of the original Fokker–Planck equation. Some examples utilizing this procedure are given in Section 3, where from the solution of the Fokker–Planck equation for symmetric stable or bistable potentials, the probability distributions for diffusion in nonsymmetric potential fields have been obtained.

2. SOLUTION OF THE FOKKER–PLANCK EQUATION FOR A BISTABLE FIELD

The Fokker–Planck equation describing the diffusion in a bistable potential field $U(x)$ is given by[1]

$$\frac{\partial P}{\partial t}=\frac{\partial}{\partial x}\left(\frac{dU}{dx}P\right)+\theta\frac{\partial^2 P}{\partial x^2} \tag{2.1}$$

where $P(x, t)$ is the probability distribution function at the time t, which is to be determined when the initial condition $P(x, t=0)$ is given. Usually it is assumed that the probability distribution is localized sharply around the point $x = y$, i.e.,

$$P(x, t=0)=\delta(x-y) \tag{2.2}$$

There are few exactly solvable cases of (2.1) with the boundary condition (2.2) which are known, and they are all related to the solvable Schrödinger equation with confining potentials.[2–7] Physically the most interesting cases of (2.1) are those where $U(x)$ is bistable and is either a symmetric or an asymmetric function of x. van Kampen has shown that the solution of (2.1) with the initial condition (2.2) can be expressed in terms of the eigenvalues of the Schrödinger equation

$$\phi_j'' + [\lambda_j - v(x)]\,\phi_j = 0, \qquad j = 0, 1, 2,... \tag{2.3}$$

provided that the ground state of (2.3) and $U(x)$ be related to each other by

$$U(x) = -2\theta \log \phi_0(x) \tag{2.4}$$

If $v(x)$ is a double well potential, then for certain values of the parameters of v, $U(x)$ will be bistable and the symmetry (or asymmetry) of $v(x)$ will determine the symmetry (or asymmetry) of $U(x)$. Supposing that all of the eigenvalues λ_j and eigenfunctions $\phi_j(x)$ of (2.3) are known, then by expanding $P(x, t)$ in terms of $\phi_j(x)\,\phi_0(x) \exp(-\lambda_j t)$ one finds that the solution of (2.1) with the initial condition (2.2) can be expressed as an infinite sum,[3]

$$P(x, y, t) = \sum_{j=0}^{\infty} \frac{\phi_0(x)}{\phi_0(y)}\,\phi_j(x)\,\phi_j(y) \exp[-\theta(\lambda_j - \lambda_0)\,t] \tag{2.5}$$

where it is assumed that the wave functions are all normalized. Due to the bistability of the potential $v(x)$, the set of eigenvalues λ_j in (2.3) satisfy the following important condition:

$$\lambda_1 - \lambda_0 \ll \lambda_j - \lambda_0, \qquad j \geqslant 2 \tag{2.6}$$

This means that after the time $t \approx [(\lambda_1 - \lambda_0)\,\theta]^{-1}$, Eq. (2.5) reduces to

$$P(x, y, t) = \phi_0^2(x) + \phi_1(x)\,\phi_1(y) \exp[-\theta(\lambda_1 - \lambda_0)\,t]\,\phi_0(x)/\phi_0(y) \tag{2.7}$$

An exponential decay law of the form (2.7) is obviously a direct result of the inequality (2.6) satisfied by the eigenvalues, and this condition in turn can be obtained by choosing $v(x)$ to be a double or a multiple well. Now let us suppose that for a certain potential $v(x)$, not necessarily double or multiple well, the Schrödinger equation is solvable and the set of eigenvalues λ_j and eigenfunctions ϕ_j are known. If these λ_j's do not satisfy the condition (2.6), we can impose (2.6) by just changing one or a few of the lowest eigenvalues. This clearly changes the potential and the eigenfunctions as well, but we do not need to find the new potential or even the new set of eigenfunctions, we can directly relate the new distribution function which we denote by $Q(x, y, t)$ to $P(x, y, t)$. Since we are changing the eigenvalues and/or the normalization of any given eigenfunction, the new set of wave functions $\psi_j(x)$ are derivable from the old set $\phi_j(x)$ by means of the Gel'fand–Levitan kernel $K(x, x')$[9,10];

$$\psi_j(x) = \phi_j(x) + \int_{-\infty}^{x} K(x, x')\, \phi_j(x')\, dx' \tag{2.8}$$

where K is the unique solution of the integral equation

$$K(x, x') + F(x, x') + \int_{-\infty}^{x} K(x, x'')\, F(x'', x')\, dx'' = 0 \tag{2.9}$$

The input function $F(x, x')$ is given in terms of the old wave functions and the normalization constants for the new wave functions $\psi_j(x)$. Confining our attention to the simplest modification of the spectrum compatible with (2.6), we change the lowest eigenvalues to ν_0 and also assume that the ground states of the new set of eigenfunctions $\psi_0(x)$ satisfy the normalization condition

$$\frac{1}{\Gamma} = \int_{-\infty}^{+\infty} \psi_0^2(x)\, dx \tag{2.10}$$

With these changes the input function $F(x, x')$ assumes the form[10]

$$F(x, x') = \Gamma \phi_0(\nu_0, x)\, \phi_0(\nu_0, x') - \phi_0(\lambda_0, x)\, \phi_0(\lambda_0, x') \tag{2.11}$$

where $\phi_0(\nu_0, x)$ is the original ground state but with λ_0 replaced by ν_0, i.e., $\phi_0(\lambda_0 \to \nu_0, x)$ Substituting (2.11) in (2.9) we observe that $K(x, x')$ is a degenerate kernel which is expressible as the sum of two terms:

$$K(x, x') = \phi_0(\nu_0, x')\, k(x) + \phi_0(\lambda_0, x')\, q(x) \tag{2.12}$$

where $k(x)$ and $q(x)$ are the solutions of a set of coupled linear algebraic equations

$$k(x) \left[1 + \Gamma \int_{-\infty}^{x} \phi_0^2(\nu_0, \xi)\, d\xi \right] + \Gamma q(x) \int_{-\infty}^{x} \phi_0(\lambda_0, \xi)\, \phi_0(\nu_0, \xi)\, d\xi + \Gamma \phi_0(\nu_0, x) = 0 \tag{2.13}$$

and

$$q(x) \left[1 - \int_{-\infty}^{x} \phi_0^2(\lambda_0, \xi)\, d\xi \right] - k(x) \int_{-\infty}^{x} \phi_0(\lambda_0, \xi)\, \phi_0(\nu_0, \xi)\, d\xi - \phi_0(\lambda_0, x) = 0 \tag{2.14}$$

Equations (2.12)–(2.14) give the explicit dependence of K on x and x', which enable us to calculate $\psi_j(x)$ from $\phi_j(x)$ for all values of j. A particularly simple case is obtained when the potential $v(x)$ is a double or multiple well and λ_j's satisfy (2.6). Then we can find a new distribution function Q from the old P, by just changing the normalization of the ground state without changing the eigenvalues. In this case $F(x, x')$ reduces to

$$F(x, x') = (\Gamma - 1)\, \phi_0(\lambda_0, x')\, \phi_0(\lambda_0, x) \tag{2.15}$$

with the corresponding $K(x, x')$;

$$K(x, x') = (1 - \Gamma)\, \phi_0(x')\, \phi_0(x) \Big/ \left[1 + (\Gamma - 1) \int_{-\infty}^{x} \phi_0^2(\xi)\, d\xi\right] \tag{2.16}$$

In general we can change the normalization of a finite number, N, of the low-lying wave functions and/or we can change a finite number, M, of the low-lying eigenvalues. The corresponding $K(x, x')$ can be obtained by solving a set of $M + N$ linear equations. For the special case of $M = N = 1$, Eq. (2.12), or $N = 1$, $M = 0$, Eq. (2.16), the completeness relation for the set of ψ_j's is given by

$$\delta(x - y) = \sum_{j=0}^{\infty} \psi_j(x)\, \psi_j(y) + (\Gamma - 1)\, \psi_0(x)\, \psi_0(y) \tag{2.17}$$

whereas for the normalized set of ϕ_j's we have

$$\sum_{j=0}^{\infty} \phi_j(x)\, \phi_j(y) = \delta(x - y) \tag{2.18}$$

Now let us assume that a solution of (2.1) with the boundary condition (2.2) is known when $U(x)$, which is given by (2.4), may or may not be bistable; we want to obtain a solution of the Fokker–Planck equation

$$\frac{\partial}{\partial t} Q(x, t) = \frac{\partial}{\partial x}\left(\frac{dW}{dx} Q\right) + \theta \frac{\partial^2 Q}{\partial x^2} \tag{2.19}$$

with the initial condition

$$Q(x, t = 0) = \delta(x - y) \tag{2.20}$$

Here $W(x)$ is related to $\psi_0(x)$ by

$$W(x) = -2\theta \log \psi_0(x) = -2\theta \log \left[\phi_0(x) + \int_{-\infty}^{x} K(x, x')\, \phi_0(x')\, dx'\right] \tag{2.21}$$

which is bistable either because $v(x)$ is a double well or because ν_0 is chosen to be very close to λ_1, i.e., $\lambda_1 - \nu_0 \ll \lambda_2 - \lambda_0$. With our choice of $W(x)$, Eq. (2.9) becomes similar to (2.1) and therefore has a solution similar to (2.5)

$$Q(x, y, t) = \sum_{j=0}^{\infty} \psi_j(x)\, \psi_j(y) \exp\,[-\theta(\lambda_j - \nu_0)\, t]\, \psi_0(x)/\psi_0(y) + (\Gamma - 1)\, \psi_0^2(x) \tag{2.22}$$

The last term, which is also a solution of (2.19), is added so that at $t = 0$, the initial condition (2.20) can be satisfied. Now $Q(x, y, t)$ can directly be related to $P(x, y, t)$ by first replacing ψ_j's by ϕ_j's using Eq. (2.8), and then summing over all j's making use of (2.18);

$$\begin{aligned}\frac{\psi_0(y)}{\psi_0(x)} Q(x, y, t) &= (\Gamma - 1)\, \psi_0(x)\, \psi_0(y) \exp[-\theta(\lambda_0 - \nu_0)\, t] \\ &\times \left\{ \frac{\phi_0(y)}{\phi_0(x)} P(x, y, t) + \int_{-\infty}^{x} K(x, x') \frac{\phi_0(x')}{\phi_0(y)} P(y, x', t)\, dx' \right. \\ &+ \int_{-\infty}^{y} K(y, y') \frac{\phi_0(y')}{\phi_0(x)} P(x, y', t)\, dy' + \int_{-\infty}^{x} \int_{-\infty}^{y} K(x, x')\, K(y, y') \\ &\left. \times \frac{\phi_0(x')}{\phi_0(y')} P(y', x', t)\, dx'\, dy' \right\}\end{aligned} \tag{2.23}$$

Note that ψ_0 and ϕ_0 are both ground-state wave functions and have no nodes therefore the integrals are all well defined. In Eq. (2.23) ψ_0 can be expressed directly in terms of ϕ_0 and $K(x, x')$ and since K depends on ν_0 and Γ, therefore the shape and the symmetry properties of ψ_0 and consequently W will be determined by these two parameters.

3. EXAMPLES

We will study two sets of examples: First those cases where $U(x)$ is bistable, and a change of normalization of the ground-state wave function is enough to obtain a new solvable Fokker–Planck equation. Consider the ground-state wave function of the double well potential[11]

$$v(x) = [\tfrac{1}{8}\xi^2 \cosh 4x - (n + 1)\, \xi \cosh 2x - \tfrac{1}{8}\xi^2] \tag{3.1}$$

which for a given integer n can be written as

$$\phi_0 = \eta_0(x) \exp(-\tfrac{1}{4}\xi \cosh 2x) \tag{3.2}$$

where $\eta_0(x)$ is a polynomial of degree n in $\cosh x$. A change in the normalization of ϕ_0 produces a new set of wave functions $\psi_j(x)$ which are related to $\phi_j(x)$ be means of (2.8), where the kernel is given by (2.16). From (2.16) and (3.2) it follows that $\psi_0(x)$ is related to $\eta_0(x)$ by

$$\psi_0(x) = \eta_0(x) \exp(-\tfrac{1}{4}\xi \cosh 2x) \Big/ \left[1 + (\Gamma - 1) \times \int_{-\infty}^{x} \eta_0^2(x') \exp(-\tfrac{1}{2}\xi \cosh 2x')\, dx' \right] \tag{3.3}$$

and hence $W(x) = -2\theta \log \psi_0(x)$ can be found. In all the cases where only the normalization of the ground-state wave function is changed, the relation between $W(x)$ and $U(x)$ takes the simple form

$$W(x) = U(x) + 2\theta \log \left\{ 1 + (\Gamma - 1) \int_{-\infty}^{x} \exp\left[-\frac{1}{\theta} U(x') \right] dx' \right\} \tag{3.4}$$

Here both U and W are bistable, but with a symmetric U, W is generally asymmetric. Next let us consider a change in the lowest eigenvalue plus a change in the normalization of the ground state. By changing the lowest eigenvalue we can adjust the long-time relaxation rate through the factors $\theta(\lambda_1 - \lambda_0)$ and $\theta(\lambda_2 - \lambda_0)$ in Eq. (2.5). The simplest example of this procedure is that of the set of wave functions for a particle in a box of lengths π. The normalized eigenvalues in this case are

$$\phi_j = \left(\frac{2}{\pi} \right)^{1/2} \sin(j+1)x, \qquad j = 0, 1, 2, \ldots \tag{3.5}$$

and therefore $P(x, y, t)$ is

$$P(x, y, t) = \sum_{j=0}^{\infty} \frac{\sin x}{\sin y} \left(\frac{2}{\pi} \right) \sin[(j+1)x] \sin[(j+1)y] \times \exp\{-\theta[(j+1)^2 - 1]\, t\} = \frac{1}{2\pi} \frac{\sin x}{\sin y} \{\theta_3[\tfrac{1}{2}(x-y), e^{-\theta t}] - \theta_3[\tfrac{1}{2}(x+y), e^{-\theta t}]\} e^{\theta t} \tag{3.6}$$

where θ_3 is the theta function.[12] From the lowest eigenfunction of (3.5), i.e., ϕ_0 which corresponds to $j = 0$ and $\lambda = 1$, we find $\phi_{\nu_0}(x)$:

$$\phi_{\nu_0}(x) = \left(\frac{2}{\pi}\right)^{1/2} \sin \nu_0^{1/2} x = \left(\frac{2}{\pi}\right)^{1/2} \sin (2 - \varepsilon)\, x \tag{3.7}$$

so that $\nu_0 = \lambda_1 - 4\varepsilon + \varepsilon^2$ with ε a small positive number. Using (3.7) and (2.11) we find $F(x, x')$:

$$F(x, x') = \frac{-2}{\pi} [\sin x \sin x' - \Gamma \sin(2 - \varepsilon)\, x \sin (2 - \varepsilon)\, x'] \tag{3.8}$$

The kernel $K(x, x')$ is degenerate and separable [Eqs. (2.13) and (2.14)] and from it we determine $\psi_0(x)$ and $W(x)$:

$$\begin{aligned} W(x) &= -2\theta \log \psi_0(x) \\ &= -2\theta \log \left\{ \left(\frac{2}{\pi}\right)^{1/2} \left[\sin x + \int_0^x K(x, x') \sin x' \, dx'\right] \right\} \end{aligned} \tag{3.9}$$

In this problem $U(x)$ is not bistable:

$$U(x) = -2\theta \log \left[(\sin x) \left(\frac{2}{\pi}\right)^{1/2} \right] \tag{3.10}$$

whereas $W(x)$ which depends on the two parameters Γ and ε can be made bistable with the appropriate choice of these two parameters.

For the third example we consider the case where ϕ_j's are harmonic oscillator wave functions, with the ground state given by

$$\phi_0(x) = \left(\frac{1}{\pi}\right)^{1/4} e^{-(1/2)x^2} \tag{3.11}$$

Since in this case ϕ_j's are solutions of the eigenvalue equation

$$\phi'' + (\lambda - x^2)\, \phi = 0 \tag{3.12}$$

for values of λ that satisfy the relation

$$\lambda_j = 2j + 1, \qquad j = 0, 1, 2,... \tag{3.13}$$

we have the harmonic oscillator wave function $\phi_j(x)$. However, when λ does not satisfy (3.13), the solution of (3.12) is given by

$$\phi(\lambda, x) = \mathcal{N} e^{-(1/2)x^2} {}_1F_1 \left(-\frac{\lambda - 1}{2}, \frac{1}{2}, x^2\right) \tag{3.14}$$

where ${}_1F_1$ is the degenerate hypergeometric function and $\mathcal{N}$ is the normalization constant. Now λ in (3.14) should be replaced by ν_0, which lies between $\lambda_0 = 1$ and $\lambda_1 = 3$, therefore we let $\nu_0 = 3 - 2\varepsilon$ with ε a number between zero and one, and with this parameter we have the normalized wave function

$$\phi_0(\nu_0, x) = (2 - 4\varepsilon + 3\varepsilon^2)^{-1/2} \left(\frac{1}{\pi}\right)^{1/4} e^{-(1/2)x^2} \times {}_1F_1[-(1-\varepsilon), \tfrac{1}{2}, x^2] \tag{3.15}$$

and hence

$$F(x, x') = \frac{1}{2 - 4\varepsilon + 3\varepsilon^2} \left(\frac{1}{\pi}\right)^{1/2} e^{-(1/2)(x^2 + x'^2)} \{1 - {}_1F_1[-(1-\varepsilon), \tfrac{1}{2}, x^2] \times {}_1F_1[-(1-\varepsilon), \tfrac{1}{2}, x'^2]\} \tag{3.16}$$

To calculate $W(x)$, we observe that $\psi_0(x)$ is given by

$$\psi_0(x) = \phi_0(\lambda_0, x) + \int_{-\infty}^{x} K(x, x')\, \phi_0(\lambda_0, x')\, dx' = \frac{N(x)}{D(x)} \tag{3.17}$$

where $K(x, x')$ is defined by Eqs. (2.12)–(2.14). Substituting for $K(x, x')$ we find

$$N(x) = \phi_0(\lambda_0, x)[1 + \Gamma \int_{-\infty}^{x} \phi_0^2(\nu_0, x')\, dx'] - \Gamma\phi_0(\nu_0, x) \int_{-\infty}^{x} \phi_0(\nu_0, x)\, \phi_0(\lambda_0, x')\, dx' \tag{3.18}$$

and

$$D(x) = 1 + \int_{-\infty}^{x} [\Gamma\phi_0^2(\nu_0, x') - \phi_0^2(\lambda_0, x')]\, dx' + \Gamma \int_{-\infty}^{x} \phi_0^2(\lambda_0, x')\, dx' \int_{-\infty}^{x} \phi_0^2(\nu_0, x')\, dx' + \Gamma \left[\int_{-\infty}^{x} \phi_0(\nu_0, x')\, \phi_0(\lambda_0, x')\, dx'\right]^2 \tag{3.19}$$

Therefore

$$W(x) = -2\theta[\log N(x) - \log D(x)] \tag{3.20}$$

4. RESULTS

Starting with a potential function for which the Schrödinger equation is solvable, we can calculate $U(x)$, the potential field in the Fokker–Planck equation, using Eq. (2.4), and determine the distribution function $P(x, y, t)$ from Eq. (2.5). We can also solve the Fokker–Planck equation (2.19) for the bistable potential field $W(x)$, where $W(x)$ is given by (2.21). The first

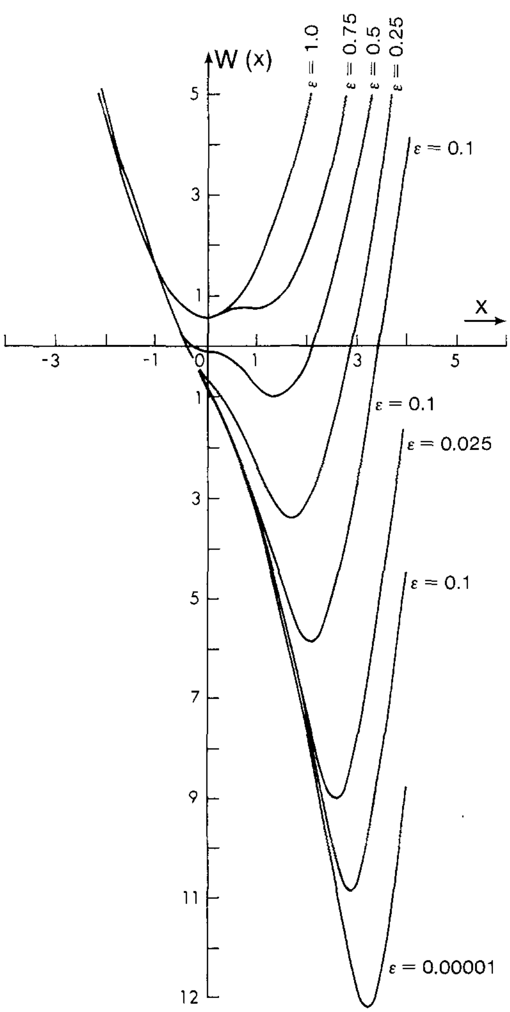

Fig. 1. The potential field $W(x)$ defined by Eqs. (2.19) and (2.21) is obtained from the solution of the harmonic oscillator problem by changing the ground-state eigenvalue [see Eqs. (3.11)–(3.19)] · $\varepsilon = 1$ refers to the oscillator eigenvalues, and $\varepsilon = 0$ corresponds to the case where the ground-state eigenvalue is deleted from the spectrum.

example that we want to discuss is the one in which $v(x)$ is the harmonic oscillator potential, i.e., $\phi_0(x)$ is given by (3.11). Here $W(x)$ can assume a bistable from if we change the lowest eigenvalue of the system. From Eqs. (3.16), (3.17), and (2.21), we find $W(x)$ which is dependent on the parameter ε, where $0 \leqslant \varepsilon \leqslant 1$. Figure 1 shows $W(x)$ for different values of ε when $\Gamma = 1$. The behavior of $W(x)$ for $\Gamma \geqslant 0.75$ is similar to that shown in Fig. 1, however for $\Gamma < 0.75$, $W(x)$ will have one minimum only for all values of ε between zero and one. We observe that in this example the height of the central maximum for $W(x)$ is very small, and this smallness of the barrier makes it difficult to present an accurate description of the different stages of the time evolution of the probability density. Therefore we consider the second example in which $v(x)$, $U(x)$, and $W(x)$ are all bistable functions and $W(x)$ is related to $U(x)$ by the change of the normalization of the ground-state wave function. Consider the case where $\phi_0(x)$ is given by (3.2) and $\psi_0(x)$ is obtained by (3.3), with the result that $W(x) = -2\theta \log \psi_0(x)$ is a bistable asymmetric potential with two well-separated minima. Here the asymmetry is determined by the parameter Γ, i.e., when $\Gamma = 1$, $W(x)$ is symmetric, but by decreasing Γ, the asymmetry becomes more pronounced. Let us consider the time evolution of the new probability density $Q(x, y, t)$ when at $t = 0$, the distribution is peaked around the central maximum of the double well potential $W(x)$, i.e.,

$$Q(x, t = 0) = \delta(x - x_m) \tag{4.1}$$

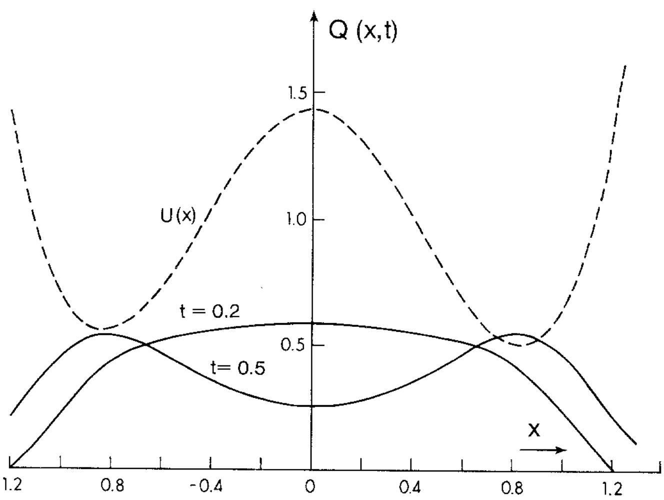

Fig. 2. A plot of $Q(x, x_m, t)$ as a function of x for two different times $t = 0.2\theta^{-1}$ and $t = 0.5\theta^{-1}$. At $t = 0, Q(x, t = 0) = \delta(x - x_m)$, where $x_m = 0$ corresponds to the macroscopically unstable point. Here the well is slightly asymmetric, $\Gamma = 0.95$, and this asymmetric well is shown by the dashed line ($\xi = 2$, $n = 3$).

where x_m lies to the left of the origin. If we choose t large enough, $Q(x, x_m, t)$ can be written as

$$Q(x, x_m, t) = \Gamma \psi_0^2(x) + \sum_{i=1}^{n+1} \psi_i(x)\, \psi_0(x) \frac{\psi_i(x_m)}{\psi_0(x_m)} e^{-\theta(\lambda_i - \lambda_0)t} \tag{4.2}$$

where n is the integer which appears in the potential $v(x)$ [Eq. (3.1)]. Since the term containing the factor $\exp[-\theta(\lambda_{n+2} - \lambda_0)\, t]$ and higher-order terms are neglected in (4.2), the values of the distribution function $Q(x, x_m, t)$ for the early stages of evolution, i.e., for times less than $[\theta(\lambda_{n+2} - \lambda_0)]^{-1}$ are not reliable.

At these early stages the initial delta function peak broadens rapidly. The broadening of the peak is followed by the appearance of two peaks, which happens after a time of the order of 0.1–$0.20\theta^{-1}$ (Figs. 2–4). Finally in the last stage the two peaks have reached their local equilibria ($t \sim 0.5\theta^{-1}$). The time-dependent probabilities of finding the Brownian particle to the left or to the right of the central maximum are given by

$$p_-(t) = \int_{-\infty}^{x_m} Q(x, x_m, t)\, dx \tag{4.3}$$

and

$$p_+(t) = \int_{x_m}^{\infty} Q(x, x_m, t)\, dx \tag{4.4}$$

These two probabilities have the long time limit of

$$p_-(\infty) = \Gamma \int_{-\infty}^{x_m} \psi_0^2(x)\, dx \tag{4.5}$$

and

$$p_+(\infty) = \Gamma \int_{x_m}^{\infty} \psi^2(x)\, dx \tag{4.6}$$

respectively. In Fig. 5, $p_-(t)$ is plotted as a function of $\log t$ ($\theta = 1$) for different values of the asymmetry parameter Γ. Note that because of terminating the infinite series (4.2) after $n + 1$ terms, for short times, the conservation of probability is violated in this approximation, i.e., $p_-(t) > 1$ at the early stages of evolution, $-3 < \log t < -1$.

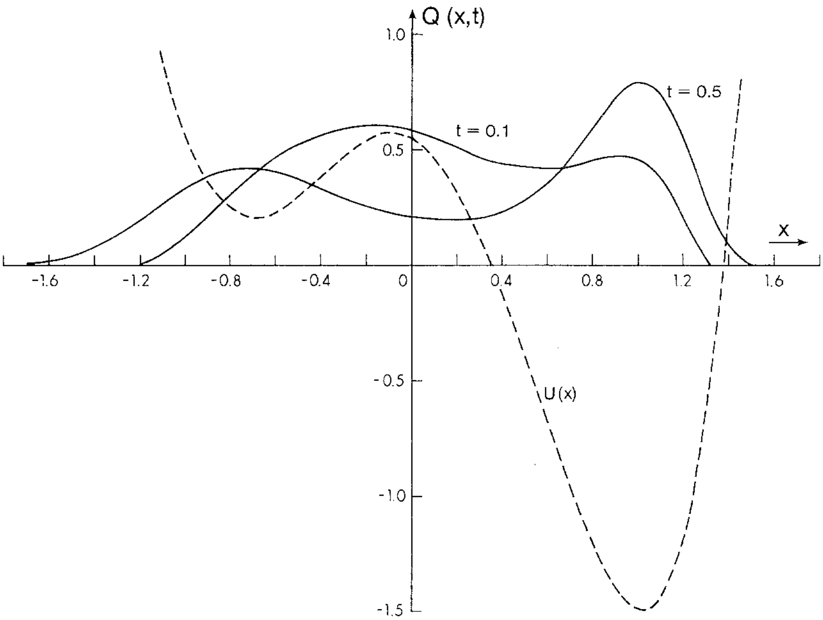

Fig. 3. $Q(x, x_m, t)$ as a function of x for $\Gamma = 0.25$ at $t = 0.2\theta^{-1}$ and $t = 0.5\theta^{-1}$. In this case $x_m = -0.1$. The dashed line shows $U(x)$ ($\xi = 2, n = 3$).

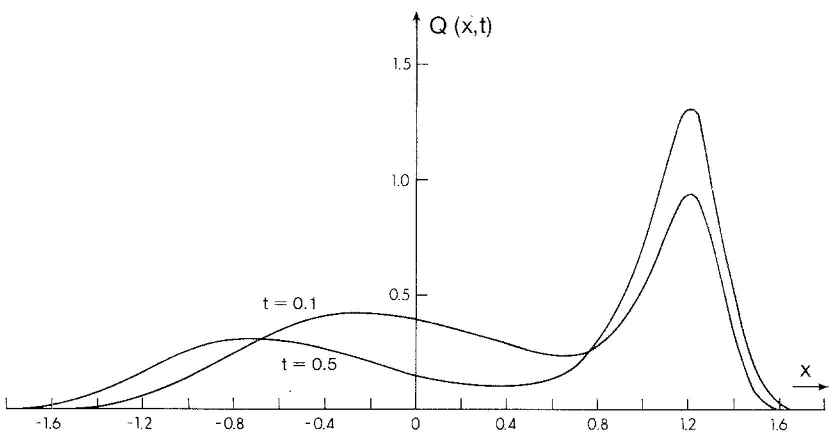

Fig. 4. Same as Fig. 3, but with $\Gamma = 0.05$ and $x_m = -0.15$.

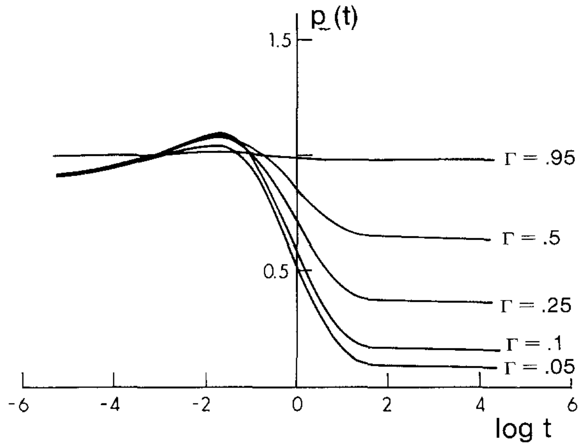

Fig. 5. The probability of finding the Brownian particle to the left of the barrier given as a function of $\log t$ ($\theta = 1$). Note that $p_-(t) > 1$ for $-3 < \log t < -1$ is the result of terminating the infinite series after three terms. $\Gamma = 1$ corresponds to a symmetric bistable potential field, Eq. (3.1), and $\Gamma = 0.05$ is a highly asymmetric case.

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